Abstract of the PhD thesis

Singular limits and rough behavior in evolutionary equations arising in physics and biology

submitted by Jakub Skrzeczkowski

The thesis discusses two topics in the theory of evolutionary equations: analysis of singular limits in PDEs and analysis of PDEs with non-standard growth where the growth changes irregularly over time. All are motivated by applications.

In the first part, we study singular limits of several PDEs from mathematical biology and physics. We begin with the fast reaction limit for a reaction-diffusion-ODE system with a nonmonotone fast reaction as motivated by applications in neuroscience. Conversely to what was observed so far for this type of problem, in the limit, we observe fast oscillations which we analyse precisely with the theory of Young measures.

Next, we study the hydrodynamic limit of the Vlasov-type equation with the appropriately chosen force so that in the limit we obtain the Cahn-Hilliard equation, a fourth-order PDE used in materials science and tumor growth. This is the first result aiming at the rigorous derivation of this macroscopic equation from the microscopic one, motivated by formal computations of Takata and Noguchi (*J. Stat. Phys.*, 2018).

Subsequently, we prove the convergence of the nonlocal Cahn-Hilliard equation to the local one. This problem has been extensively studied in recent years. Our work is the first one to consider degenerate mobilities as motivated by applications to tumor growth and cell adhesion. It can be viewed as a completion of the Giacomin-Lebowitz derivation of the Cahn-Hilliard equation from particle processes on the lattice (*J. Stat. Phys.*, 1997): they derived the nonlocal equation and left it open to prove its convergence to the local equation. This is the gap we fill with our results.

Finally, we discuss the convergence of the Euler-Korteweg equation to the Cahn-Hilliard equation in the so-called high-friction limit. This problem was studied recently by Lattanzio and Tzavaras (*Comm. PDEs*, 2017) who proved, using the rela-

tive entropy method, the convergence under the assumption that the limiting system admits a smooth positive solution. However, there is no theory guaranteeing that. Therefore, we propose to study the nonlocal Euler-Korteweg equation. Then, the limiting system is the nonlocal Cahn-Hilliard equation which has the desired properties and we can conclude by the relative entropy method. Furthermore, using the result of nonlocal-to-local convergence for the Cahn-Hilliard equation, we obtain convergence of the nonlocal Euler-Korteweg equation to the local Cahn-Hilliard equation.

The second part of the thesis is concerned with parabolic PDEs of non-standard growth where the growth is changing discontinuously in time. The classical example is the $p(t,x)$-Laplace equation with p strictly separated from 1 and $+\infty$. We prove existence and uniqueness of solutions for p discontinuous in t and log-Hölder continuous in x. This is the first result of this type as all the papers so far assumed log-Hölder continuity in both t and x. The proof is based on a simple observation that mollification in space of a solution to a parabolic equation is already regular in time.

Subsequently, we extend the existence result to the case of non-Newtonian fluids with stress tensor which is discontinuous in time. This is well-motivated by behavior of electrorheological fluid (a fluid composed of charged particles) moving in the electric field which drastically changes in time.

Finally, we briefly report on our result on double phase functionals, that is functionals switching their growth between p and q, depending on the point of the space. This is a thoroughly studied topic since the groundbreaking work of Mingione and Colombo (*ARMA*, 2014). Using methods we developed, we improve so-far known range of exponents p, q such that the minimizers can be approximated in a nice way (so-called lack of Lavrentiev phenomenon). In the case $p \leq d$ (d is the dimension), we obtain the first sharp result concerning the range of exponents (A. K. Balci et al, *Calc. Var. PDE*, 2020). This is important as it is usually the first step to prove the smoothness of the minimizers. In applications, such minimizers describe the optimal configuration of a composite material under an external force.